§4.2. $\iint_D f\,dA$

1–6 Evaluate the iterated integral.

#3. $\int_0^1 \left(\int_y^{e^y} \sqrt{x}\,dx\right)dy = \int_0^1 \left(\frac{2}{3}x^{\frac{3}{2}}\Big|_{x=y}^{x=e^y}\right)dy$

$= \frac{2}{3}\int_0^1 \left(e^{\frac{3}{2}y} - y^{\frac{3}{2}}\right)dy = \frac{2}{3}\left[\frac{2}{3}e^{\frac{3}{2}y}\Big|_{y=0}^{y=1} - \frac{2}{5}y^{\frac{5}{2}}\Big|_{y=0}^{y=1}\right]$

$= \frac{4}{9}e^{\frac{3}{2}} - \frac{32}{45}$ ∎

#6. $\int_0^1 \left(\int_0^v \sqrt{1-v^2}\,du\right)dv = \int_0^1 \left(\sqrt{1-v^2}\,u\Big|_{u=0}^{u=v}\right)dv$

$= \int_0^1 v\sqrt{1-v^2}\,dv = \int_0^1 (1-v^2)^{\frac{1}{2}}\,\frac{dv^2}{11}\cdot\frac{1}{2}$

$= -\frac{1}{2}\cdot\frac{2}{3}\,(1-v^2)^{\frac{3}{2}}\Big|_{v=0}^{v=1} = \frac{1}{3}$ ∎

$d(1-v^2)\,(-1)$

7–16 Evaluate the double integral.

#9. $\iint_D \frac{2y}{x^2+1}\,dA$, $D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le \sqrt{x}\}$

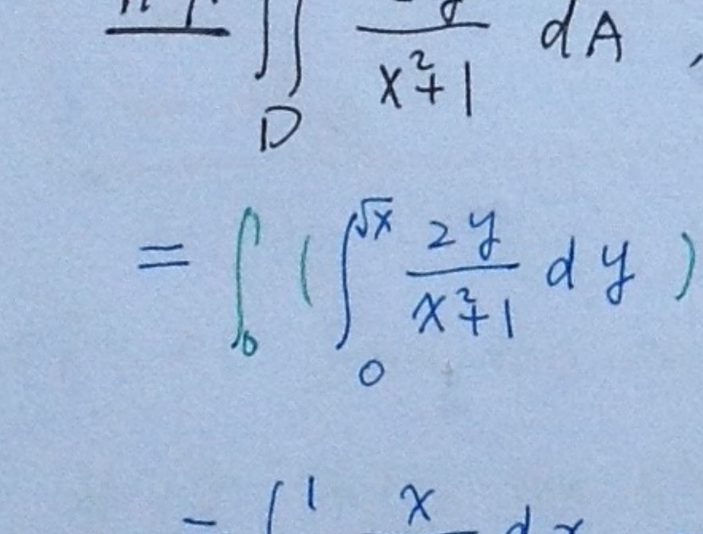
↑ 先積

$= \int_0^1 \left(\int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\right)dx = \int_0^1 \left(\frac{y^2}{x^2+1}\Big|_{y=0}^{y=\sqrt{x}}\right)dx$

$= \int_0^1 \frac{x}{x^2+1}\,dx = \int_0^1 (x^2+1)^{-1}\,\frac{dx^2}{11}\cdot\frac{1}{2} = \frac{1}{2}\ln(x^2+1)\Big|_{x=0}^{x=1} = \frac{\ln 2}{2}$

$d(x^2+1)$ ∎

#13. $\iint_D y^3\,dA$, D is the triangular region with vertices $(0,2)$, $(1,1)$, $(3,2)$.

1° 

D: $\begin{cases} 1 \le y \le 2 \\ 2-y \le x \le 2y-1 \end{cases}$

过 $(1,1)$ $(0,2)$ L: $x+y=2 \Rightarrow x = 2-y$

过 $(1,1)$ $(3,2)$ L: $2y = x+1 \Rightarrow x = 2y-1$

2° $\iint_D y^3\,dA$

$= \int_1^2 \left(\int_{2-y}^{2y-1} y^3\,dx\right)dy$

$= \int_1^2 (3y^4 - 3y^3)\,dy$

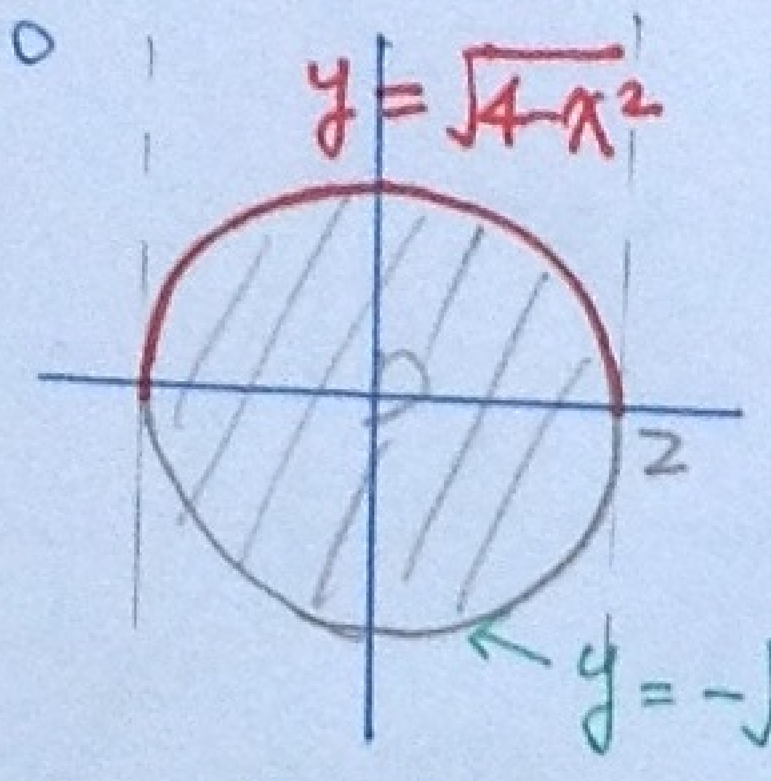
$= \frac{3}{5}y^5 - \frac{3}{4}y^4\Big|_1^2 = \frac{147}{20}$ ∎

§4.2 $\iint f\, dA$

#15. $\iint_D (2x-y)\,dA$, D is bdd. by the circle with center 0 and $r=2$

• 没使用对称性要花点時間才做得出来!!

1°



$x^2+y^2=4$

$y^2=4-x^2$

$D:\begin{cases} -2\le x\le 2 \\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\end{cases}$

2° $\iint_D (2x-y)\,dA$

odd w.r.t $y \Rightarrow \int_{-a}^{a} y\,dy=$

$= \int_{-2}^{2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\,dy\right)dx$

or 在此亦看出 = 0

$= \int_{-2}^{2}\left(2xy-\frac{1}{2}y^2 \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right)dx$ (the $\frac{1}{2}y^2$ term crossed out → 0)

$= \int_{-2}^{2} 4x\sqrt{4-x^2}\,dx = 0$

odd. w.r.t x (let $f(x)=4x\sqrt{4-x^2}$

則 $f(-x)=-4x\sqrt{4-x^2}=-f(x)$)

或直接计算

$= 2\int_{-2}^{2}(4-x^2)^{\frac{1}{2}}\,dx^2.$

$= -2\int_{-2}^{2}(4-x^2)^{\frac{1}{2}}\,d(4-x^2) = -\frac{4}{3}(4-x^2)^{\frac{3}{2}}\Big|_{x=-2}^{x=2} = 0$ #

x=2 代入之值与

x=-2 代入之值一样

一减就 = 0 了.

> 〈另法〉使用 symmetric
>
> 因 D 对 x=0 及 y=0 均对称.
>
> 又 2x is odd in x
>
> and y is odd in y
>
> $\Rightarrow \iint_D (2x-y)\,dA = 0$ #

17–26. Find the volume of solid

#17 Under the plane $x+2y-z=0$ and above the region bdd by $y=x$ and $y=x^4$.

↑ $z=x+2y$ 是屋顶 (integrand)

sol: 1° 只需画 D

$\begin{cases} y=x \\ y=x^4\end{cases} \Rightarrow x(x^3-1)=0$

$\Rightarrow x=0$ or 1

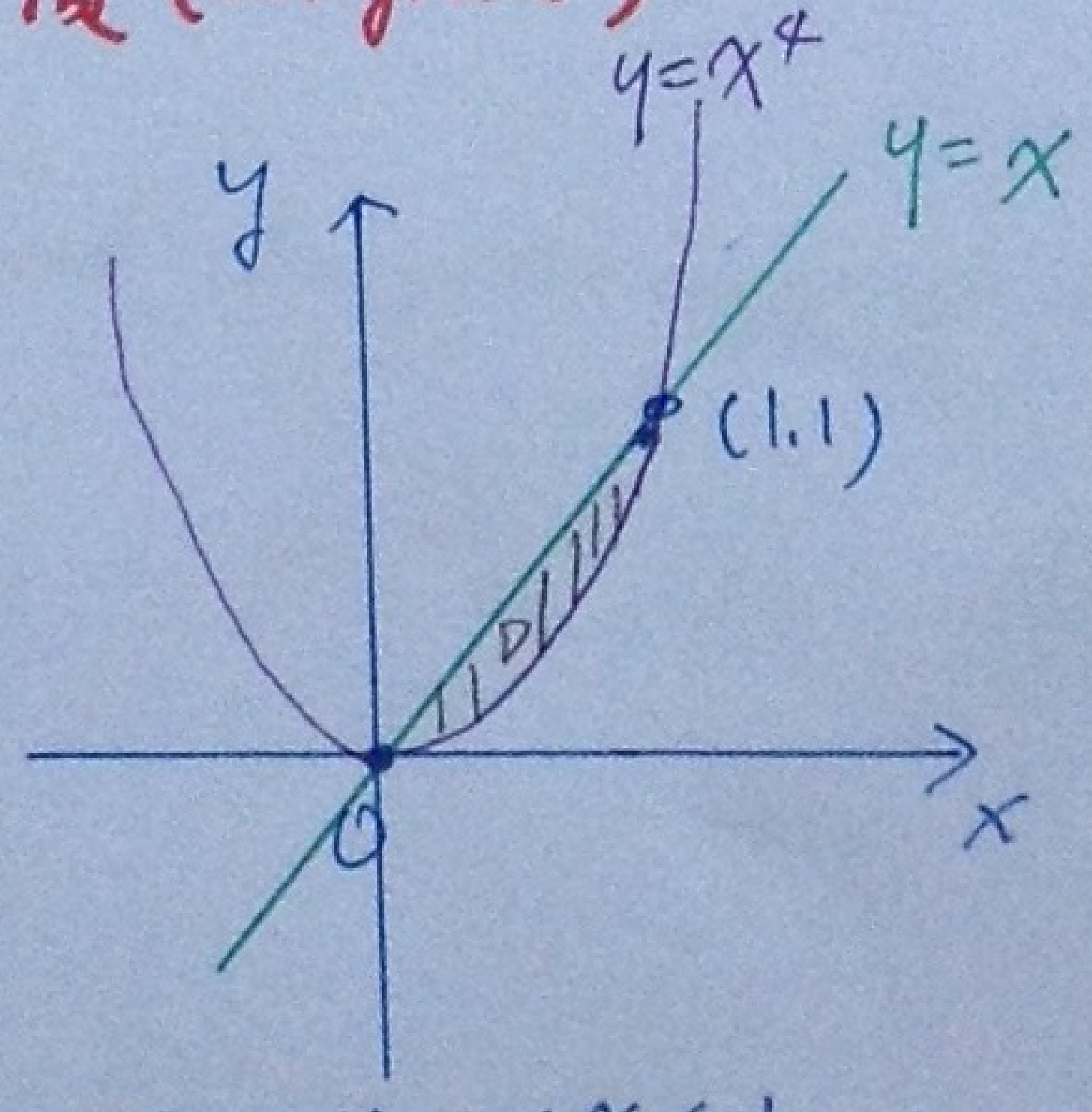
↓ ↓

$y=0$ $y=1$

有两交点

(0, 0)

(1, 1)



$D:\begin{cases} 0\le x\le 1 \\ x^4\le y\le x\end{cases}$

2° $V=\iint_D (x+2y)\,dA$

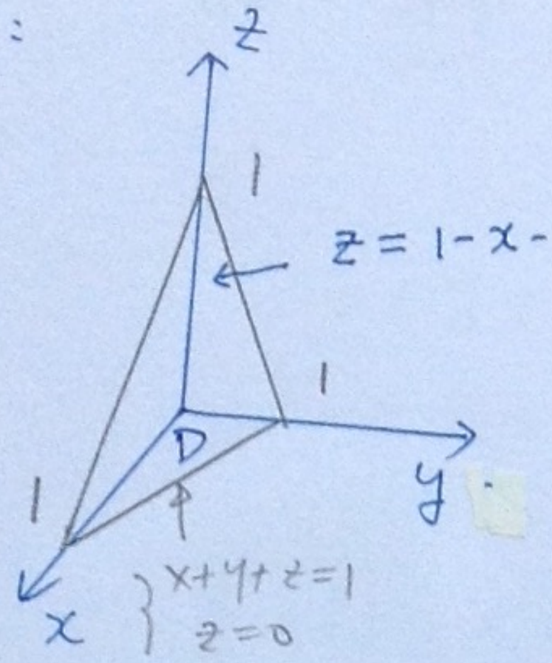
$=\int_0^1\left[\int_{x^4}^{x}(x+2y)\,dy\right]dx$

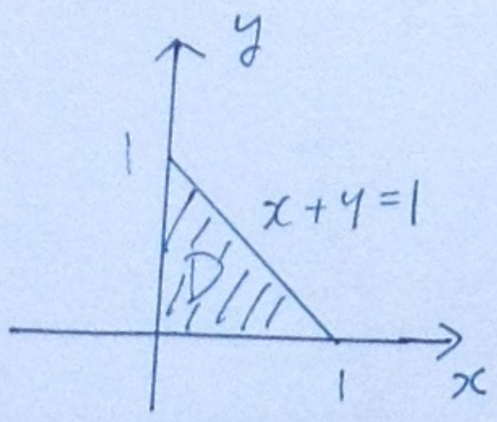
$=\int_0^1 (2x^2-x^5-x^8)\,dx = \frac{2}{3}x^3-\frac{1}{6}x^6-\frac{1}{9}x^9\Big|_0^1 = \frac{7}{18}$ #

§4.2 $\iint f\,dA$

#21. Bdd. by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$

sol:





$$D=\begin{cases}0\le x\le 1\\ 0\le y\le 1-x\end{cases}$$

$$V=\iint_D (1-x-y)\,dA$$

$$=\int_0^1\left(\int_0^{1-x}(1-x-y)\,dy\right)dx$$

$$=\int_0^1 y-xy-\frac{y^2}{2}\Big|_{y=0}^{y=1-x}\,dx$$

$$=\int_0^1 \frac{1}{2}(1-x)^2\,dx$$

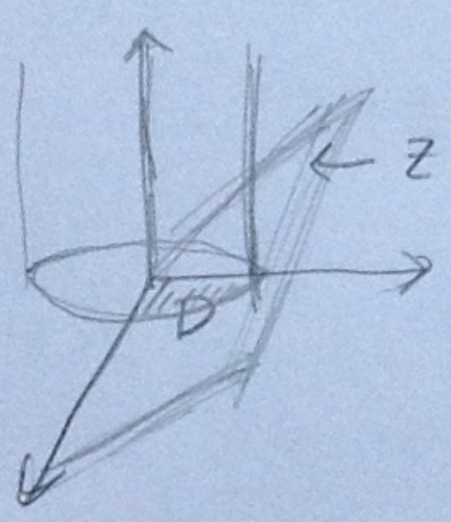
$$=-\frac{1}{6}(1-x)^3\Big|_{x=0}^{x=1}=\frac{1}{6}$$ ※

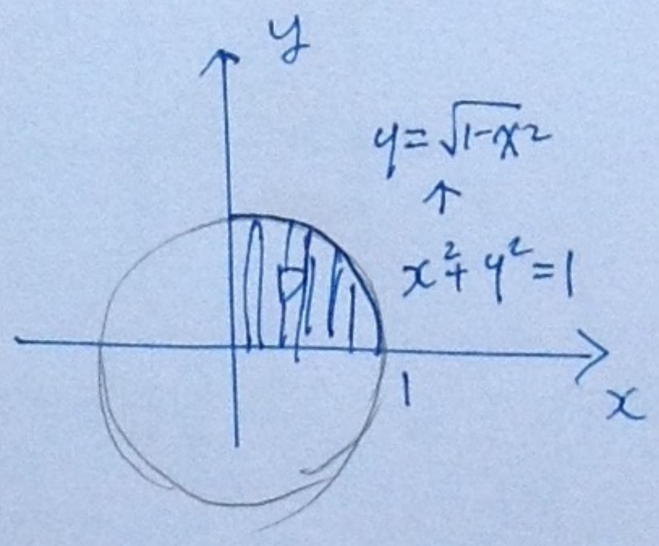
另法：$V=\frac{1}{3}\times$底面積$\times$高

$$=\frac{1}{3}\times\left(\frac{1}{2}\times1\times1\right)\times1=\frac{1}{6}$$ ※

#25. Bdd by the cylinder $x^2+y^2=1$ and the planes $y=z$, $x=0$, $z=0$ in the first octant.

sol:





$$D=\begin{cases}0\le x\le 1\\ 0\le y\le\sqrt{1-x^2}\end{cases}$$

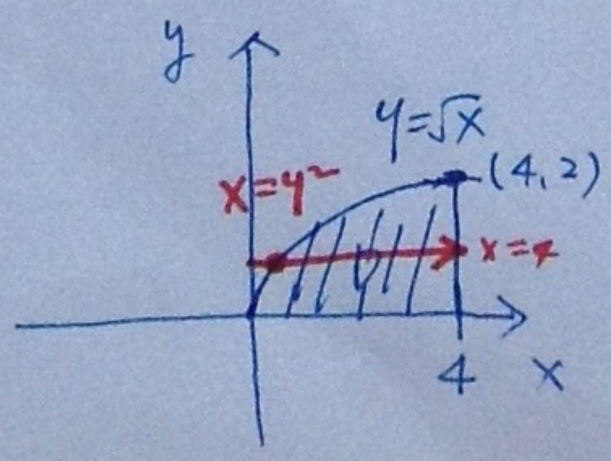
$$V=\iint_D y\,dA=\int_0^1\left(\int_0^{\sqrt{1-x^2}} y\,dy\right)dx=\int_0^1\frac{1}{2}(1-x^2)\,dx$$

$$=\frac{1}{2}x-\frac{1}{6}x^3\Big|_0^1=\frac{1}{3}$$ ※

31-36 sketch the region of integration and change the order of integration.

#31 $\int_0^4\int_{\frac{x}{2}}^{\sqrt{x}} f(x,y)\,dy\,dx=\iint_D f(x,y)\,dA=\int_0^2\int_{y^2}^{4} f(x,y)\,dx\,dy$

sol: $D=\begin{cases}0\le x\le 4\\ \frac{x}{2}\le y\le\sqrt{x}\end{cases}$



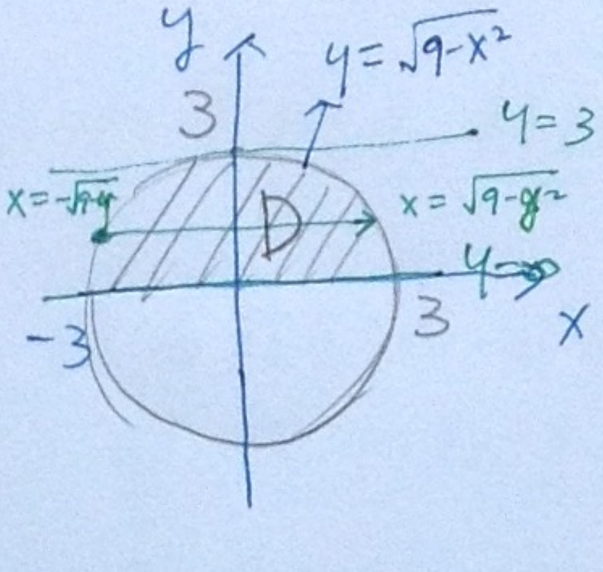
$D=\begin{cases}0\le y\le 2\\ y^2\le x\le 4\end{cases}$

§4.2 $\iint_D f\,dA$

#33. $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dA = \int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} f(x,y)\,dy\,dx$

Sol:

$$\begin{cases} 0 \le y \le 3 \\ -\sqrt{9-y^2} \le x \le \sqrt{9-y^2} \end{cases}$$



$$D: \begin{cases} -3 \le x \le 3 \\ 0 \le y \le \sqrt{9-x^2} \end{cases}$$

圓 bdy $x^2=9-y^2$

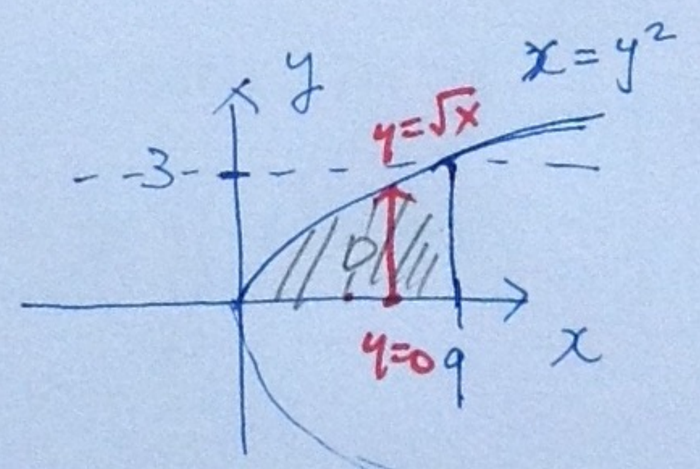
$x^2+y^2=3^2$

37-42. Evaluate the integral by reversing the order of integration

#39 $\int_0^3 \int_{y^2}^{9} y\cos(x^2)\,dx\,dy$

Sol:

$$D: \begin{cases} 0 \le y \le 3 \\ y^2 \le x \le 9 \end{cases}$$



$$D: \begin{cases} 0 \le x \le 9 \\ 0 \le y \le \sqrt{x} \end{cases}$$

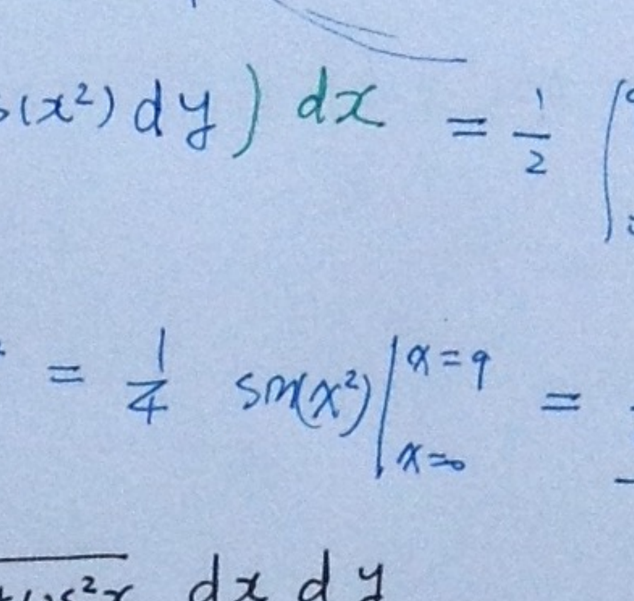
原式 $= \int_0^9 \left(\int_0^{\sqrt{x}} y\cos(x^2)\,dy \right) dx = \frac{1}{2} \int_0^9 x\cos(x^2)\,dx$

$= \frac{1}{4} \int_0^9 \cos(x^2)\,dx^2 = \frac{1}{4} \sin(x^2)\Big|_{x=0}^{x=9} = \underline{\frac{1}{4}\sin 81}$ #

#41 $\int_0^1 \int_{\sin^{-1}y}^{\frac{\pi}{2}} \cos x \sqrt{1+\cos^2 x}\,dx\,dy$

Sol:

$$D: \begin{cases} 0 \le y \le 1 \\ \sin^{-1}y \le x \le \frac{\pi}{2} \end{cases}$$



$$D: \begin{cases} 0 \le x \le \frac{\pi}{2} \\ 0 \le y \le \sin x \end{cases}$$

圓 $\sin^{-1}y = x$

即 $y = \sin x$

原式 $= \int_0^{\frac{\pi}{2}} \left(\int_0^{\sin x} \cos x \sqrt{1+\cos^2 x}\,dy \right) dx$

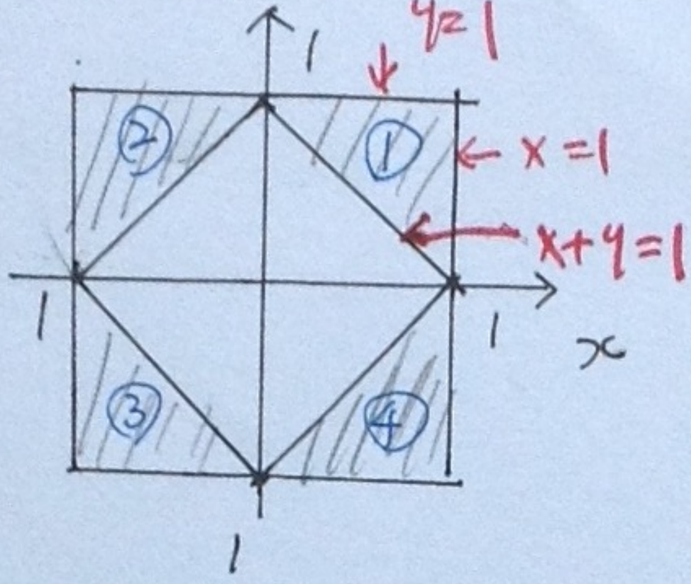
$= \int_0^{\frac{\pi}{2}} \sin x \cos x \sqrt{1+\cos^2 x}\,dx = -\int_0^{\frac{\pi}{2}} \cos x (1+\cos^2 x)^{\frac{1}{2}}\,d\cos x$

§4.2 $\iint_D f\,dA$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos^2 x)^{\frac{1}{2}}\, d(1+\cos^2 x)$$

$$= -\frac{1}{2}\cdot\frac{2}{3}(1+\cos^2 x)^{\frac{3}{2}}\Big|_{x=0}^{x=\frac{\pi}{2}} = \frac{1}{3}[2\sqrt{2}-1]$$ ※

#43. $\iint_D x^2\,dA$，其中 D 如图所示斜线部分区域



因右边区域 ① ∪ ④ 与左边区域 ② ∪ ③ 对称 y-axis (ie $x=0$)

又 $f(x,y)=x^2$ 对 x 是 even

$$\therefore \iint_D x^2\,dA = 2\iint_{D_{右}} x^2\,dA \qquad D_{右} = ① \cup ④$$

又 ① 与 ④ 对称 x-axis (ie $y=0$)

且 $f(x,y)=x^2$ 是 even in y

$$\therefore \iint_{D_{右}} x^2\,dA = 2\iint_{①} x^2\,dA$$

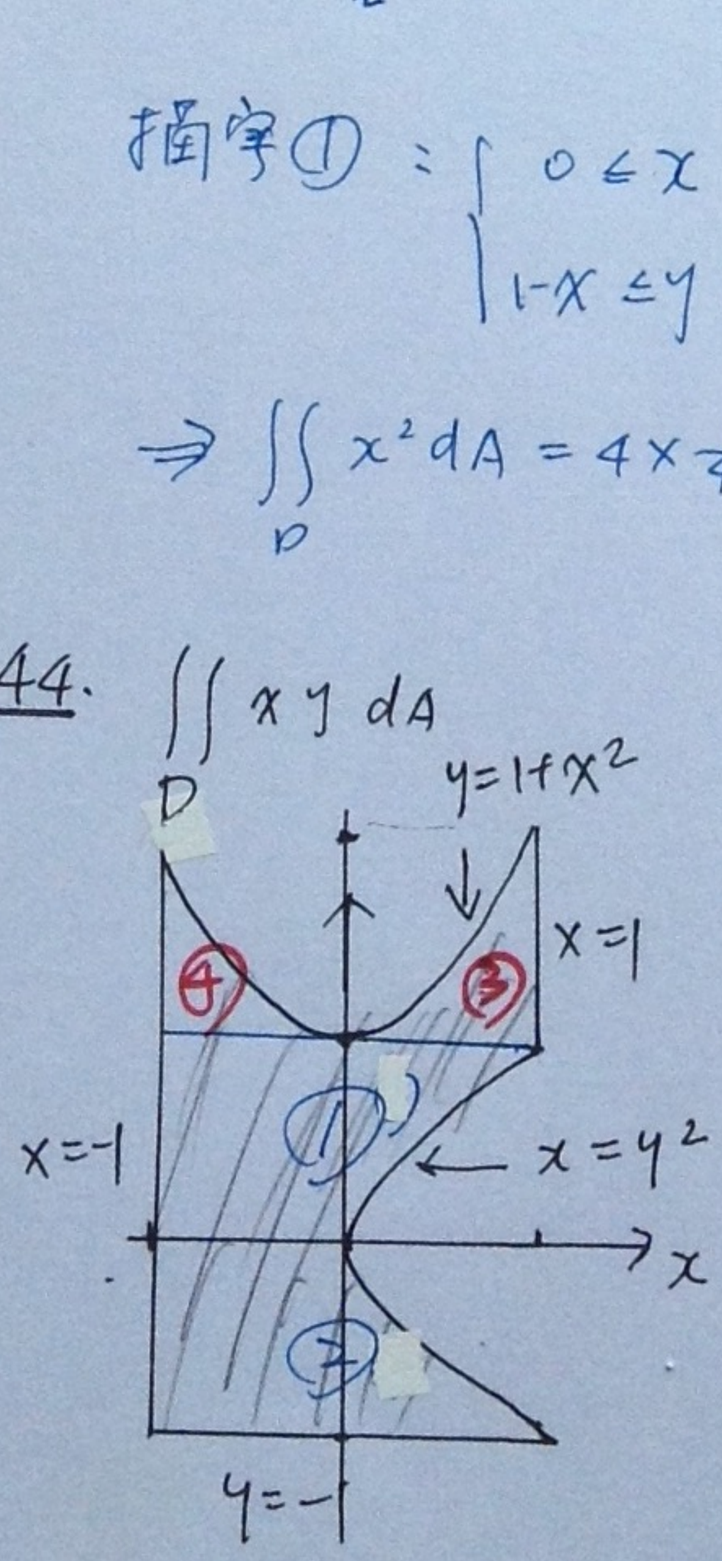
Summary:
$$\iint_D x^2\,dA = 4\iint_{①} x^2\,dA$$

描字①：$\begin{cases} 0\le x\le 1 \\ 1-x\le y\le 1 \end{cases}$ $\Rightarrow \iint_{①} x^2\,dA = \int_0^1 \left(\int_{1-x}^1 x^2\,dy\right) dx$

$$= \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}$$

$$\Rightarrow \iint_D x^2\,dA = 4\times\frac{1}{4} = 1$$ ※

#44. $\iint_D xy\,dA$



〈法一〉分割成 $D=D_1\cup D_2$ 来做

上 下

⇑ ⇑

I II

〈法二〉使用对称性来做

因函数 $f(x,y)=xy$ 是 odd in x

and 是 also odd in y

① 与 ② sym $y=0$ $\Rightarrow \iint_{①\cup②} xy\,dA = 0$

③ 与 ④ sym $x=0$ $\Rightarrow \iint_{③\cup④} xy\,dA = 0$

$$\Rightarrow \iint_D xy\,dA = \iint_{①\cup②} + \iint_{③\cup④} = 0$$ ※